Black Hole Area Quantization rule from Black Hole Mass Fluctuations

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In the early days of Black Hole Thermodynamics, Bekenstein calculated the black hole mass dispersion that arises from the random emission of quanta that satisfy either Bose-Einstein or Fermi-Dirac statistics [1]. In that paper Bekenstein concluded that the mass dispersion becomes negative for black holes of masses larger than $\sim 10^{19} g$ and named it *the mass spread paradox*. In this paper we retake that calculation obtaining the mass distribution probability for a given averaged black hole mass. From this result, we obtain the black hole's mass dispersion and show that, as expected, there is no mass spread paradox. Then, from a complete different perspective we deal with the black hole as an atom with a quantized event horizon area. Following Bekenstein [2] we postulate transition probabilities among the various energy levels and then calculate the mass dispersion that arises from this quantum mechanical behavior. It turns out that there is a perfect agreement between the phenomenological and the microscopic calculations if and only if the area spectrum is linear, in agreement with the quasi-normal-modes , the adiabatic invariant Sommerfeld quantization and loop quantum gravity. Accordingly, the quantum mechanical properties of the black hole which are supposedly relevant only at Planckian scales do leave an imprint in the black hole mass dispersion at much larger scales where gravity can be dealt classically.

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I. PHENOMENOLOGICAL CALCULATION OF THE BLACK HOLE MASS DISPERSION

Black hole evaporation is a stochastic process – fluctuations of the thermal radiation emitted as the black hole evaporates imply on a fluctuation of the black hole mass. Assume a mass distribution probability [1]

$$P(M, \langle M \rangle) \tag{1}$$

which is the probability of having a black hole of mass M knowing that its average mass is $\langle M \rangle$. We shall assume that bh's average mass $\langle M \rangle$ is very large and that the probability function peaks for some value of the same order as $\langle M \rangle$. Then we can approximate the probability function as a continuous function of the mass M. Clearly :

$$\int_0^\infty P(M, \langle M \rangle) = 1 \tag{2}$$

$$\langle M \rangle \equiv \int_0^\infty M P(M, \langle M \rangle) \tag{3}$$

$$\Sigma^2 \equiv \int_0^\infty (M - \langle M \rangle)^2 P(M, \langle M \rangle) \tag{4}$$

Following Bekenstein's paper, consider a black hole of mass M^* which by Hawking's radiation [3] decays into a black hole of mass M; the difference $\Delta M = M^* - M$ is the energy carried out by the radiation $\sum_i \epsilon_i n_i$, where ϵ_i is the energy of a given field mode and n_i the number of particles emitted in that mode. Let $p_i(\epsilon_i)$ represent the emission probability for that mode. Clearly after this emission, the new probability function is

$$P(M, \langle M \rangle) = \sum_{\{n_i\}} P(M + \sum_i \epsilon_i n_i, \langle M^* \rangle) \prod_i p_i(\epsilon_i)\,. \tag{5}$$

Expanding up to the second order in the emitted energy:

$$P(M, \langle M \rangle) = P(M, \langle M^* \rangle) + \frac{\partial P(M, \langle M^* \rangle)}{\partial M} \sum_i \sum_{n_i} \epsilon_i n_i p(n_i) + \frac{1}{2} \frac{\partial^2 P(M, \langle M^* \rangle)}{\partial M^2} \left[\sum_i \sum_{n_i} \epsilon_i^2 n_i^2 p(n_i) + \sum_{i \neq j} \epsilon_i \epsilon_j \sum_{n_i, n_j} n_i n_j p(n_i) p(n_j) \right] \tag{6}$$

Since

$$\sum_i \sum_{n_i} \epsilon_i n_i p(n_i) = \sum_i \epsilon_i \overline{n}_i\,, \tag{7}$$

$$\sum_i \sum_{n_i} \epsilon_i^2 n_i^2 p(n_i) = \sum_i \epsilon_i^2 \overline{n_i^2}\,, \tag{8}$$

and, furthemore

$$\sum_{i \neq j} \sum_{n_i, n_j} \epsilon_i \epsilon_j n_i n_j p(n_i) p(n_j) = \sum_{i \neq j} \epsilon_i \epsilon_j \overline{n}_i \overline{n}_j = \sum_i \epsilon_i \overline{n}_i \left(\sum_j \epsilon_j \overline{n}_j - \epsilon_i \overline{n}_i \right)\,, \tag{9}$$

It follows that

$$P(M, \langle M \rangle) - P(M, \langle M^* \rangle) = \frac{\partial P(M, \langle M^* \rangle)}{\partial M} \overline{\Delta M} + \frac{1}{2} \left[\sigma^2 + \overline{\Delta M}^2 \right] \frac{\partial^2 P(M, \langle M^* \rangle)}{\partial M^2} \tag{10}$$

where

$$\overline{\Delta M} \equiv \sum_i \epsilon_i \overline{n}_i \tag{11}$$

and

$$\sigma^2 \equiv \sum_i \epsilon_i^2 \left(\overline{n_i^2} - \overline{n}_i^2 \right)\,, \tag{12}$$

correspond to the mean energy of the radiation emitted by the black hole and its (squared) fluctuation. In order to keep the discussion simple, assume that the black hole temperature is such that only photons are emitted. The mean number of emitted particles and the squared number deviation are given by:

$$\overline{n_i} = \frac{1}{e^{\alpha_i} - 1} \tag{13}$$

$$\overline{n_i^2} - \overline{n_i}^2 = \frac{e^{\alpha_i}}{(e^{\alpha_i} - 1)^2} \tag{14}$$

where the parameter α is fixed via the average number of emitted particles in Hawking's radiation:

$$\frac{1}{e^{\alpha_i} - 1} \equiv \frac{\Gamma_i}{e^{8\pi M \epsilon_i} - 1}\,, \tag{15}$$

where Γ_i is the mode dependent black hole absorptivity. For massless radiation this absorptivity depends upon $M\epsilon_i$ alone. Two words of caution. First, there is an ambiguity in the above identification of α_i as the rhs in eq.(15) can be either identified with the black hole's actual mass M or its averaged value $\langle M\rangle$. There is no a priori reason to prefer one of these possibilities over the other. Secondly, there are two different averaging procedures, the one accounting for fluctuations on the number of emitted quanta which is calculated from the Bose-Einstein and Fermi-Dirac statistics and is hereby represented by an 'overbar'; the average with respect to the black hole fluctuations is calculated from the probability function $P(M, \langle M\rangle)$ and is denoted by a pair of brackets.

Multiplying eq (10) by M and integrating over the mass, gives

$$-\Delta\langle M\rangle \equiv \langle M\rangle - \langle M^*\rangle = \int_0^\infty \overline{\Delta M} M \frac{\partial P(M, \langle M^*\rangle)}{\partial M} dM + \frac{1}{2}\int_0^\infty \left[\sigma^2 + \overline{\Delta M}^2\right] M \frac{\partial^2 P(M, \langle M^*\rangle)}{\partial M^2} dM\,. \tag{16}$$

Should we calculate α in eq. (15) identifying M with the averaged mass $\langle M\rangle$ then both $\overline{\Delta M}$ and σ^2 depend only on $\langle M\rangle$ (see eqs.(11)–(15)) and can be pulled out in both integrals. After integration by parts

$$\Delta\langle M\rangle = \overline{\Delta M}\,, \tag{17}$$

were we assumed that the probability distribution is such that $P(M, \langle M\rangle)$; $MP(M, \langle M\rangle)$ and its derivative all vanish at both ends of the integral. Similarly multiplying eq.(10) by M^2 integrating by parts and further assuming that M^2P vanishes as $M \to \infty$, if follows that

$$\overline{M^2} - \overline{M^{*2}} = \sigma^2 + \overline{\Delta M}^2 - 2\langle M\rangle\overline{\Delta M}\,. \tag{18}$$

With the aid of the identity:

$$\overline{M^2} - \overline{M^{*2}} = \Sigma^2 - \Sigma^{*2} - 2\langle M\rangle\Delta\langle M\rangle + (\Delta\langle M\rangle)^2\,, \tag{19}$$

and in virtue of eq.(17), we obtain the relation

$$\Sigma^2(\langle M\rangle) - \Sigma^2\left(\langle M\rangle - \overline{\Delta M}\right) = \sigma^2\,. \tag{20}$$

Whenever the mass $M >> M_p$, then $\overline{\Delta M} << \langle M\rangle$, it is legitimate to expand the lhs to the first order and consequently

$$\frac{d\Sigma^2}{d\langle M\rangle} = \frac{d\overline{\Delta M}}{d\langle M\rangle}\frac{\sigma^2}{\overline{\Delta M}}\,. \tag{21}$$

Multiplying eqs. (11) and (12) by M and M^2, respectively and taking the continuous limit, integrating over $x \equiv \langle M\rangle\epsilon$ it follows that $\langle\Delta M\rangle \sim \langle M\rangle^{-1}$ and $\sigma^2 \sim \langle M\rangle^{-2}$ the black hole mass dispersion reads

$$\Sigma^2 \sim \frac{m_p^4}{\langle M\rangle^2}\,. \tag{22}$$

Let us now see what would happen should M stand for its face value in eq. (15) . Then, going to the continuous limit with $x \equiv M\epsilon_i$, if follows that $\overline{\Delta M} \sim M^{-1}$ and $\sigma^2 \sim M^{-2}$. We can no longer pull out this quantities from the integrals as before. Multiply eq. (10) by M and integrate. The first integral in eq.(10) vanishes and assuming that P/M^2 and P'/M vanish at both ends of the integral, by integration by parts the second integral gives

$$\Delta\langle M\rangle \sim -m_p^4\langle\frac{1}{M^3}\rangle \tag{23}$$

which is clearly unacceptable, the evaporation does not increase the black hole mass. Thus the right procedure is to identify M in eq.(15) with the black hole averaged mass $\langle M\rangle$.

II. A MICROSCOPICAL VIEW: A BLACK HOLE AS AN ATOM

Black hole quantization area follows from very different arguments, ranging from quantization of the adiabatic quantity [7], the gedanken experiment of critical absorption of a quanta [5], loop quantum gravity[8] and quantization of the black hole normal modes [6]. Let us write

$$A(n) = 16\pi G M^2 = \chi m_p^2 n \tag{24}$$

where χ is a dimensionless constant and we assumed a non-rotating black hole. The transition of the black hole to lower level $n \to n-1$ entails the emission of a quanta of energy $\hbar\omega$:

$$\hbar\omega / T_{bh} = 4\chi \tag{25}$$

where $T_{bh} = \frac{\hbar}{8\pi M G}$ is the Bekenstein-Hawking temperature. As long as $n >> 1$ the black hole temperature remains very nearly constant and the energy levels are equally spaced. Following Bekenstein [2], the probability for a downwards transition to a contiguous level $p(n \to n-1) = e^{-\beta}$ corresponds to a one-step emission of a quantum, very much like in atomic physics. Similarly, $p(n \to n+1) = e^{-\mu}$ corresponds to a upwards transition to a contiguous state with the absorption of a quanta. From the detailed balance condition of a black hole in thermal equilibrium with radiation it follows that $e^{-\mu} = e^{-\beta} e^{x}$, which by its turn requires that the degeneracy of each level to be $g(n) = e^{S_{bh}}$, consistent with the interpretation of the microcanonical entropy as the number of available black hole states. In the pure evaporation process of a black hole we are interested in transitions from the n to a $m < n$ level. Accordingly, in a $n \to m$ transition in a cascading process

$$p(n \to m) = (1 - e^{-\beta}) e^{-\beta(n-m)} \tag{26}$$

where $(1 - e^{-\beta})$ corresponds to a no-emission probability. Although the probabilities are not strictly normalized

$$1 - \sum_{m=0}^{n} p(n \to m) = e^{-\beta(n+1)} \tag{27}$$

for large n's the difference is exponentially small. Thus, up to an exponentially small error we can extend the sum to $n \to \infty$. The black hole mass fluctuation is

$$\Sigma^2 = \langle M^2 \rangle - \langle M \rangle^2 = \frac{\chi m_p^2}{16\pi} \left(\langle m \rangle - \langle \sqrt{m} \rangle^2 \right) . \tag{28}$$

We proceed by calculating

$$\langle m \rangle = \sum_{m=0}^{n} m\, p(n \to m) = (1 - e^{-\beta}) \sum_{p=0}^{\infty} (n-p) m e^{-\beta p} = n - \frac{1}{e^{\beta} - 1} . \tag{29}$$

Similarly

$$\langle \sqrt{m} \rangle = \sum_{m=0}^{\infty} \sqrt{m}\, p(n \to m) = \sqrt{n} (1 - e^{-\beta}) \sum_{p=0}^{\infty} \sqrt{1 - \frac{p}{n}} e^{-\beta p} . \tag{30}$$

Recalling the expansion

$$\sqrt{1-x} = 1 - \frac{1}{2} x - \frac{1}{8} x^2 + \ldots , \tag{31}$$

then up to the second order

$$\langle \sqrt{m} \rangle = \sqrt{n} (1 - e^{-\beta}) \left[\sum_{p=0}^{\infty} e^{-\beta p} - \frac{1}{2n} \sum_{p=0}^{\infty} p e^{-\beta p} - \frac{1}{8n^2} \sum_{p=0}^{\infty} p^2 e^{-\beta p} + \cdots \right] , \tag{32}$$

or

$$\langle \sqrt{m} \rangle = \sqrt{n} \left[1 - \frac{1}{2n} \frac{1}{e^{\beta} - 1} - \frac{1}{8n^2} \frac{e^{\beta} + 1}{(e^{\beta} - 1)^2} + \cdots \right] . \tag{33}$$

Finally

$$\langle\sqrt{m}\rangle^2 = \left[n - \frac{1}{e^\beta - 1} - \frac{1}{4n}\frac{e^\beta}{(e^\beta - 1)^2} + \mathcal{O}(\frac{1}{n^2})\cdots\right]. \tag{34}$$

In view of eqs.(29) and (34), it follows that

$$\Sigma^2 = \frac{\chi}{64\pi n}\frac{e^\beta}{(e^\beta - 1)^2} + \mathcal{O}(\frac{1}{n^2}) \tag{35}$$

Thus, to the leading order

$$\Sigma^2 = \frac{\chi^2}{2^9\pi^2}\frac{m_p^4}{\langle M\rangle^2}\frac{e^\beta}{(e^\beta - 1)^2}. \tag{36}$$

which is identical to eq.(22) that was obtained from purely statistical considerations. We remark that the discrete linear area spectrum is the only one which is consistent with phenomenological derivation. Therefore the quantum mechanical behavior of the black hole in quantum regime where black hole area quantization becomes relevant leaves a fingerprint in the large mass scale limit where gravity can be certainly treated as a classical field. We remind that the present calculation rests upon elementary probability theory, the Hawking radiation and furthermore, that this radiation results from transitions among various black hole quantum states which is a feature valid for any quantum mechanical system. Accordingly, this is a very robust and model independent result that tells us that the radiation that follows from transitions at the black hole quantum theory is related to the radiation of massive black holes that can be dealt within the framework of theB classical theory of gravity . This is nothing but a manifestation of the correspondence principle for black holes, a principle that played such a fundamental role as a guiding tool in the early development of quantum mechanics. As a concluding remark, the phenomenological calculation tells that Σ^2 must be a linear function of the number of species the black hole radiates at a given mass scale [see eq.(21)].According, so must χ depend on the number of particle species, which means that Quantum Gravity must be intertwined with the Quantum Theory of Matter fields.

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